REFERENCES

1. BOGDANOVA E.V. and RYZHOV O.S., On perturbations generated by oscillators in a viscous fluid flow at post-critical frequencies. Prikl. Mekh. Tekh. Fiz., 4, 1982.
2. TERENT'EV E.D., The linear problem of a vibrator performing harmonic oscillations at beyond-critical frequencies in a subsonic boundary later, PMM, 48, 2, 1984.
3. NEILAND V.YA., Asymptotic problems of the theory of viscous supersonic flows. Trudy, TsAGI, 1529, 1974.
4. STEWARTSON K., Multistructured boundary layers on flat plates and related bodies. Advances in Applied Mechanics, 14, Academic Press, N.Y., 1974.
5. RUBAN A.I. and SYCHEV V.V., Asymptotic theory of laminar boundary layer separation in an incompressible fluid, Uspekhi Mekhaniki, 2, 4, 1979.
6. RYZHOV O.S. and TERENT'EV E.D., On the non-stationary boundary layer with selfinduced pressure, PMM, 41, 6, 1977.
7. RYZHOV O.S. and ZHUK V.I., Internal waves in the boundary layer with selfinduced pressure, J. Mec., 19, 3, 1980.
8. TERENT'EV E.D., The linear problem of a vibrator in a subsonic boundary layer, PMM, 45, 6, 1981.
9. RYZHOV O.S. and TERENT'EV E.D., On certain properties of vortex spots in the boundary layer on a plate. Dokl. Akad. Nauk SSSR, 276, 3, 1984.
10. GASTER M. and GRANT I., An experimental investigation of the formation and development of a wave packet in a laminar boundary layer. Proc. Roy. Soc. A, 374, 1649, 1975.
11. ZHUK V.I. and RYZHOV O.S., Free interaction and stability of the boundary layer in an incompressible fluid, Dokl. Akad. Nauk SSSR, 253, 6, 1980.
12. ZHUK V.I. and RYZHOV O.S., On the asymptotic of solutions of the Orr-Sommerfeld equation that yield unstable oscillations for large values of the Reynolds number, Dokl. Akad. Nauk SSSR, 268, 6, 1983.
13. ZHUK V.I., On the asymptotic form of solutions of the Orr-Sommerfeld equations in domains adjoining two branches of the neutral curve, Izv. Akad. Nauk SSSR, Mekhan. Zhidk. Gaza, 4, 1984.
14. LIN C.C., Theory of Hydrodynamic Stability, IIL, Moscow, 1958.

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REMOTE STATIONARY WAVE FIELD GENERATED BY LOCAL PERTURBING SOURCES IN A FLOW OF STRATIFIED FLUID*

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A linear formulation is used to study the problem of stationary waves formed in a uniform flow of an inviscid incompressible vertically stratified fluid past a point source or a mass dipole. Formulas are derived representing the characteristics of the wave field in the form of the sum of single integrals. A method is developed for constructing complete asymptotic expansions of the integrals obtained for large distances from the wave generator, including uniform expansions near the leading fronts of the separate modes. Approximate solutions of the problem in question exist (/1-4/ et al.). The behaviour of the characteristics of the wave field near the leading fronts of internal waves was studied in /5, 6/. In the case of a deep liquid the asymptotic form uniform in the neighbourhood of the leading fronts is expressed in terms of Fresnel integrals /5/, and in the case of a liquid of finite depth by Airy functions /6/. Examples of the exact solution of the problem are given in /7/.

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1. Let an inviscid incompressible fluid occupy the region $-\infty < x, y < +\infty$, $-h < z < 0$ and let it flow with constant velocity c in the positive direction of the horizontal x axis. The velocity of the unperturbed fluid $\rho_0(z)$ depends on the single vertical z coordinate, the function $\rho_0(z)$ is monotonic and non-increasing. A point source of constant intensity q is situated at the depth h_1 from the level of the unperturbed free surface $z = 0$ of the fluid. The stationary wave field generated by the source is described in the linear approximation by the equation for the vertical component of velocity $w(x, y, z)$ /2, 3/ with the boundary conditions

$$D(\rho_0\,\partial w/\partial z) + \rho_0(N^2c^{-2} + \partial^2/\partial x^2)\,\Delta_2 w = qD[\rho_0\delta(x, y, z + h_1)] \tag{1.1}$$

$$(D - gc^{-2}\Delta_2)\,w = 0\ (z = 0),\ w = 0\ (z = -h) \tag{1.2}$$

$$w \to 0\ (x^2 + y^2 \to \infty)$$

$$D = \partial^3/\partial x^2\partial z,\ \ \Delta_2 = \partial^2/\partial x^2 + \partial^2/\partial y^2,\ \ N^2(z) = -g\rho_0^{-1}\partial\rho_0/\partial z$$

Here $N(z)$ is the Brunt-Väisälä frequency, g is the acceleration due to gravity, and $\delta(\cdot)$ is the delta function. The boundary conditions must be supplemented with the radiation condition, which is that the main wave perturbations are formed in the downstream direction.

Applying to (1.1) and (1.2) a Fourier transformation in the variables x and y we obtain, for the transform of the vertical component of velocity,

$$W(r, \theta, z) = (2\pi)^{-1}\iint_{-\infty}^{\infty} w(x, y, z)\exp[-ir(x\cos\theta + y\sin\theta)]\,dx\,dy$$

$$0 \leqslant r < \infty,\ 0 \leqslant \theta < 2\pi$$

the following boundary value problem:

$$(\rho_0 W_z)_z + \rho_0[N^2(c\cos\theta)^{-2} - r^2]\,W = (2\pi)^{-1}q\,[\rho_0\delta(z + h_1)]_z \tag{1.3}$$

$$W_z - g(c\cos\theta)^{-2}\,W = 0\ (z = 0),\ W = 0\ (z = -h)$$

Let us choose $\beta = r^2$ as the spectral parameter, and let β_n, W_n $(n = 1, 2, \ldots;\ \beta_1 > \beta_2 > \ldots)$ be a set of eigenvalues and orthonormalized eigenfunctions of the Sturm-Liouville problem

$$(\rho_0 W_z)_z + \rho_0(N^2\lambda - \beta)\,W = 0\ \ (-h < z < 0) \tag{1.4}$$

$$W_z - g\lambda W = 0\ \ (z = 0),\ \ W = 0\ \ (z = -h),\ \ \lambda = (c\cos\theta)^{-2}$$

$$\left(\int_{-h}^{0}\rho_0 W_n W_m\,dz = 0,\ \ n \neq m;\ \ \int_{-h}^{0}\rho_0 W_n^2\,dz = 1\right)$$

The solution of the inhomogeneous boundary value problem (1.3) can be written in the form of a series /2, 3, 7/

$$W = (2\pi)^{-1}q\rho_0(-h_1)\sum_{n=1}^{\infty}\Phi_n(z, -h_1, \theta)(r^2 - \beta_n)^{-1} \tag{1.5}$$

$$\Phi_n(z, -h_1, \theta) = W_n(z, \theta)\,W_{nz}(-h_1, \theta)$$

Let us list briefly the basic properties of the dispersion relations $\beta_n(\theta)$, known from the work done concerning the internal waves excited by the moving sources of perturbations, or the waves in the moving fluid excited by the stationary sources. Let us write $0 \leqslant \theta < \pi/2$. The properties $\beta_n(\theta)$ can be easily transformed to other intervals of variation in θ, since the functions $\beta_n(\theta)$ depend in fact on $\lambda = (c\cos\theta)^{-2}$. Thus the values of $\beta_n(\theta)$ are real; the functions $\beta_n(\theta)$ increase monotonically and tend to infinity as $\theta \to \pi/2$. If $c < c_n$ (c_n is the velocity of propagation of the longitudinal waves of the n-th mode), then $\beta_n(\theta) > 0$. If $c > c_n$, then the function $\beta_n(\theta)$ has a single simple zero $\theta = \theta_n$, $\theta_n = \arccos(c_n/c)$. We know /1/ that when $c > c_n$, the angular width of the region of basic wave perturbations of the n-th mode is equal to $2\arcsin(c_n/c)$.

Applying to (1.5) an inverse Fourier transformation, we obtain an expression for the vertical component of the velocity in the form of a sum of double integrals

$$w(x, y, z) = (2\pi^2)^{-1}q\rho_0(-h_1)\sum_{n=1}^{\infty} w_n(x, y, z) \tag{1.6}$$

$$w_n(x, y, z) = \operatorname{Re}\int_{-\pi/2}^{\pi/2}\Phi_n(z, -h_1, \theta)\,I_n(\theta, R, \gamma)\,d\theta \tag{1.7}$$

$$I_n(\theta, R, \gamma) = \int_0^{\infty} r(r^2 - \beta_n)^{-1}\exp[irR\cos(\theta - \gamma)]\,dr \tag{1.8}$$

Here R, γ are polar coordinates in the horizontal $(x, y): x = R\cos\gamma,\ y = R\sin\gamma$-plane. When $\beta_n > 0$, the path of integration in (1.8) passes the pole along the small semicircle in the lower complex half-plane of the parameter r. Such a method of bypassing the singularity is determined when considering the problem of establishing the wave field /7/.

The vertical displacements of the fluid particles in the stream $\zeta(x, y, z)$, caused by the action of the source are connected, in the linear approximation, with the vertical component of the velocity, by the kinematic relation $c\,\partial\zeta/\partial x = w$. Integrating (1.6)-(1.8) in x, we obtain

$$\zeta(x, y, z) = q(2\pi^2 c)^{-1}\rho_0(-h_1)\sum_{n=1}^{\infty}\zeta_n(x, y, z) \tag{1.9}$$

$$\zeta_n(x, y, z) = \operatorname{Re}\int_{-\pi/2}^{\pi/2}\Phi_n(z, -h_1, \theta)\cos^{-1}\theta J_n(\theta, R, \gamma)\,d\theta \tag{1.10}$$

$$J_n(\theta, R, \gamma) = -i\int_0^{\infty}(r^2 - \beta_n)^{-1}\exp[irR\cos(\theta - \gamma)]\,dr \tag{1.11}$$

The integration path in (1.11) is the same as in (1.8).

The variable change $r = |\beta_n|^{1/2}t$ enables us to collect all the parameters occurring in the integrands in (1.8) and (1.4) in a single combination $R|\beta_n|^{1/2}\cos(\theta - \gamma)$. Regarding J_n and I_n as Laplace transforms and using the known analytic properties of the latter, we obtain

$$I_n = G(-R\Delta_n),\quad J_n = r_n^{-1}F(-R\Delta_n),\quad \Delta_n(\theta, \gamma) = r_n(\theta)\cos(\theta - \gamma) \tag{1.12}$$

Here $G(p)$ and $F(p)$ are the analytic continuations of the functions

$$g(p) = \int_0^{\infty} t(t^2 + 1)^{-1}e^{-pt}\,dt,\quad f(p) = \int_0^{\infty}(t^2 + 1)^{-1}e^{-pt}\,dt\quad (\operatorname{Re} p > 0)$$

in the complex plane of the parameter p with a cut $(-\infty, 0]$; $r_n(\theta) = \beta_n^{1/2}(\theta)$ if $\beta_n > 0$ and $r_n(\theta) = i[-\beta_n(\theta)]^{1/2}$ if $\beta_n < 0$ the root is arithmetic. We note that when θ varies from $-\pi/2$ to $\pi/2$, the curve $p = -R\Delta_n(\theta, \gamma)$ moves only along the axes of the complex p plane. The edge of the cut is chosen arbitrarily when $\operatorname{Im}\Delta_n = 0,\ \operatorname{Re}\Delta_n > 0$, since we need only the real parts of the functions $G(p)$ and $F(p)$ in formulas (1.7) and (1.10), while $\operatorname{Re} G(p)$ and $\operatorname{Re} F(p)$ vary continuously in the course of the passage through $(-\infty, 0]$.

2. Let us describe briefly the properties of the functions $G(p)$ and $F(p)$ which will be needed when analysing the solutions obtained.

1°. We have the following formulas /8/ in the region $|\arg p| < \pi$:

$$G(p) = -\operatorname{Ci}(p)\cos p + \operatorname{si}(p)\sin p,\quad F(p) = \operatorname{Ci}(p)\sin p + \operatorname{si}(p)\cos p \tag{2.1}$$

($\operatorname{si}(p) = \pi/2 - \operatorname{Si}(p)$; $\operatorname{Si}(p)$ and $\operatorname{Ci}(p)$ are the integral sine and cosine respectively).

2°. Information on the behaviour of the functions $G(p)$ and $F(p)$ near the point $p = 0$ is given by the expressions following from (2.1) and from the definitions of $\operatorname{Si}(p)$ and $\operatorname{Ci}(p)$ /8/

$$G(p) = -\ln p\cos p + g_1(p),\quad F(p) = \ln p\sin p + f_1(p) \tag{2.2}$$

where $g_1(p)$ and $f_1(p)$ are entire functions, and the principal branch in the complex p plane with a cut $(-\infty, 0]$ is chosen for $\ln p$.

3°. Differentiating (2.1) we can confirm that

$$G(p) = -\frac{d}{dp}F(p),\quad F(p) = \frac{d}{dp}[G(p) + \ln p] \tag{2.3}$$

4°. When $|p| \to \infty$, we have the asymptotic expansions /8/

$$G(p) \sim \sum_{m=0}^{\infty}(-1)^m(2m + 1)!\,p^{-2m-2},\quad F(p) \sim \sum_{m=0}^{\infty}(-1)^m(2m)!\,p^{-2m-1} \tag{2.4}$$

The expansions are uniform in $\arg p$ when $|\arg p| \leqslant \pi - \varepsilon,\ \varepsilon > 0$.

5°. From the definition of the functions $G(p)$ and $F(p)$ it follows that

$$G(-p) = G(p) + is\pi e^{isp},\quad F(-p) = -F(p) + \pi e^{isp},\quad s = \operatorname{sign}(\arg p) \tag{2.5}$$

Using (2.4) and (2.5), we can construct an asymptotic form of the functions $G(p)$ and $F(p)$ as $|p| \to \infty$, uniform in the neighbourhood of the cut.

Thus the integrals (1.8) and (1.11) have been expressed in terms of known functions, and we can assume that the solutions obtained for the vertical displacement and vertical velocity fields generated by a point source are sums of single integrals of the form

$$w_n = \mathrm{Re}\int_{-\pi/2}^{\pi/2} \Phi_n G(-R\Delta_n)\,d\theta, \tag{2.6}$$

$$\zeta_n = \mathrm{Re}\int_{-\pi/2}^{\pi/2} \Phi_n (r_n\cos\theta)^{-1} F(-R\Delta_n)\,d\theta$$

3. Let us now analyse the contribution of the n-th mode in the remote region of the wave field (as $R \to +\infty$). We exclude from our discussion the neighbourhood of the vertical plane $y = 0$. We assume that $\delta \leqslant \gamma \leqslant \pi - \delta$, δ is a small positive number. First we will make certain comments concerning the construction of asymptotic expansions of the integrals (2.6). Formula (2.2) shows that the integrands in (2.6) have logarithmic singularities at points which are the zeros of Δ_n. The function $\Delta_n(\theta, \gamma) = r_n(\theta)\cos(\theta - \gamma)$ has a zero $\theta = \theta_0$, $\theta_0 = \gamma - \pi/2$ and, provided that $c > c_n$, $\theta = \pm\theta_n$ are also the zeros of $r_n(\theta)$. The case when $c = c_n$ when $\theta_n = 0$ is not discussed. If $\theta_0 \neq \pm\theta_n$, then θ_0 is a simple zero; $d\Delta_n(\theta_0, r)/d\theta = r_n(\theta_0) \neq 0$ and the points $\pm\theta_n$ are zeros with multiplicity of $\frac{1}{2}$:

$$d\Delta_n^2(\pm\theta_n, \gamma)/d\theta = \cos^2(\pm\theta_n - \gamma)\,d\beta_n(\pm\theta_n)/d\theta \neq 0$$

Let us use the expansion of the unity /9/

$$1 \equiv \eta(\theta) + \eta(\theta, \theta_0) + \eta(\theta, \theta_n) + \eta(\theta, -\theta_n) \tag{3.1}$$

Here $\eta(\theta, \tau)$ is an infinitely differentiable finite function, different from zero only in a certain neighbourhood $U(\tau)$ of the point $\theta = \tau$, $\eta(\tau, \tau) = 1$, $d^m\eta(\tau, \tau)/d\theta^m = 0$, $m \geqslant 1$. The function $\eta(\theta)$ complements to unity in the interval $(-\pi/2, \pi/2)$ the sum of the remaining three terms.

In accordance with (3.1) the integrals (2.6) decompose into a sum of four integrals, three of which represent the contributions of the zeros of Δ_n. The carrier of the function $\eta(\theta)$ represents the union of intervals (two when $c < c_n$ and four when $c > c_n$) on which the quantity $|\Delta_n|$ has a uniform lower limit. According to the theorem on the integration of asymptotic series /9/ the formulas for w_n and ζ_n, asymptotic as $R \to +\infty$, can be written using (2.4) and (2.5) in the form of a power series and a Fourier integral. The asymptotic form of the Fourier integral is equal, with the accuracy of up to $O(R^{-\infty})$, to the sum of the contributions of the boundary points and stationary points /9/. The boundary points of the integrals in question are the zeros of Δ_n and $\theta = \pi/2$. The contribution of the zeros of Δ_n is already separated by the introduction of the functions $\eta(\theta, \tau)$, and the point $\theta = \pi/2$ yields a contribution towards the integrals (2.6) of the order $O(R^{-\infty})$ when $\delta \leqslant \gamma \leqslant \pi - \delta$. The stationary points θ_k are solutions of the equations $d\Delta_n/d\theta = 0$ under the condition that $\mathrm{Re}\Delta_n > 0$, $\mathrm{Im}\Delta_n = 0$. The number of stationary points $N(\gamma)$ depends on the parameter γ. It can be shown that $N(\gamma)$ is always finite.

Thus the above analysis shows that the asymptotic form of the integrals (2.6) has, with the accuracy of up to $O(R^{-\infty})$, the form

$$\begin{bmatrix} w_n \\ \zeta_n \end{bmatrix} \sim S_{n\nu}(R) + \sum_{\tau=\theta_0, \pm\theta_n} Z_{n\nu}(R, \tau) + \sum_{k=1}^{N(\gamma)} D_{n\nu}(R, \theta_k) \tag{3.2}$$

$$S_{n\nu}(p) \sim \sum_{m=0}^{\infty} (-1)^{m+\nu}(2m+1-\nu)!\,R^{-\nu-2m-2} \times \tag{3.3}$$

$$\int_{-\pi/2}^{\pi/2} \Phi_{n\nu}\eta\Delta_n^{\nu-2m-2}\,d\theta, \quad \Phi_{n\nu} = \Phi_n(r_n\cos\theta)^{-\nu}$$

Here $\nu = 0$ for w_n and $\nu = 1$ for ζ_n. The contribution of the stationary point θ_k is the integral

$$D_{n\nu}(R, \theta_k) = \pi\,\mathrm{Re}\,[i^{(1-\nu)}\int_{-\pi/2}^{\pi/2} \Phi_{n\nu}\eta(\theta, \theta_k)\exp(iR\Delta_n)\,d\theta] \tag{3.4}$$

and the contribution of the zero τ and of the function Δ_n is the integral

$$Z_{n\nu}(R, \tau) = \mathrm{Re}\int_{-\pi/2}^{\pi/2} \Phi_{n\nu}\eta(\theta, \tau)\,E_\nu(-R\Delta_n)\,d\theta \tag{3.5}$$

$$(E_0(p) = G(p), \quad E_1(p) = F(p))$$

The formulas which give full asymptotic expansions of the contributions of the simple, near and multiple stationary points can be found in /9/.

We obtain the asymptotic forms of the integrals (3.5) by integration by parts with the help of (2.3). The first integration yields

$$Z_{n\nu}(R,\tau)=\nu R^{-1}\int\limits_{-\pi/2}^{\pi/2} M[\Phi_{n\nu}\eta(\theta,\tau)]\ln|\Delta_n|\,d\theta -$$

$$(-1)^{\nu}R^{-1}\int\limits_{-\pi/2}^{\pi/2} M[\Phi_{n\nu}\eta(\theta,\tau)]\,E_{1-\nu}(-R\Delta_n)\,d\theta,$$

$$M[\Phi]=\left(\frac{\Phi}{\Delta_{n\theta}'}\right)_{\theta}'$$

The second integral in the above formula has the same form as the initial integral. Integrating it by parts we obtain the term of the order $O(R^{-2})$, etc. As a result we have

$$Z_{n\nu}(R,\tau)\sim\sum_{m=1}^{\infty}(-1)^{m+\nu}R^{\nu-2m}\int\limits_{-\pi/2}^{\pi/2} M^{2m-\nu}[\Phi_{n\nu}\eta(\theta,\tau)]\ln|\Delta_n|\,d\theta \tag{3.6}$$

We note that the asymptotic forms of the contributions of the zeros in θ of the function $\Delta_n(\theta,\gamma)$, represents power series similar to (3.3). Let

$$S_{n\nu}^{1}(R)=S_{n\nu}(R)+\sum_{\tau=\theta_0,\,\pm\theta_n} Z_{n\nu}(R,\tau)$$

Adding the series for $S_{n\nu}$ and $Z_{n\nu}$ term by term, we obtain

$$S_{n\nu}^{1}(R)\sim\sum_{m=0}^{\infty}(-1)^{m+\nu}(2m+1-\nu)!\,\alpha_{nm}(\nu)\,R^{\nu-2m-2} \tag{3.7}$$

$$\alpha_{nm}(\nu)=\int\limits_{-\pi/2}^{\pi/2}\Phi_{n\nu}\eta(\theta)\,\Delta_n^{\nu-2m-2}\,d\theta - \tag{3.8}$$

$$\frac{1}{(2m+1-\nu)!}\sum_{\tau=\theta_0,\,\pm\theta_n}\int\limits_{-\pi/2}^{\pi/2} M^{2m+2-\nu}[\Phi_{n\nu}\eta(\theta,\tau)]\ln|\Delta_n|\,d\theta$$

It can be confirmed that (3.8) represents a regularized form of the integral

$$\alpha_{nm}(\nu)=\int\limits_{-\pi/2}^{\pi/2}\Phi_n(r_n\cos\theta)^{-\nu}\,\Delta_n^{\nu-2m-2}\,d\theta \tag{3.9}$$

in which $\Delta_n^{\nu-2m-2}$ must be regarded as a generalized function. In particular,

$$\alpha_{n0}(1)=\text{v. p.}\int\limits_{-\pi/2}^{\pi/2}\Phi_n[r_n^2\cos\theta\cos(\theta-\gamma)]^{-1}\,d\theta$$

Thus the asymptotic forms of the integrals (2.6) are represented by the sums of the power series (3.7) and contributions of the stationary points. The first term of the power series for w_n in $O(R^{-2})$, and for ζ_n it is $O(R^{-1})$.

4. The asymptotic expansions obtained will not be uniform in γ: $\delta\leqslant\gamma\leqslant\pi-\delta$, $\delta>0$ when $\theta_0\to-\theta_n$ or $\theta_0\to\theta_n$. When $\theta_0\to-\theta_n-0$, the zeros of Δ_n merge with the stationary point situated between them. The vertical plane in the (x, y, z) space for which $\theta_0=-\theta_n$ or $\gamma=\gamma_n$, $\gamma_n=\arcsin c_n/c$, corresponds to the boundary of the region of the basic wave perturbations of the n-th node. In the case when $\gamma>\gamma_n$, the equation of stationary phase $d\Delta_n/d\theta=0$ has no solutions. When $\theta_0\to-\theta_n+0$ or $\theta_0\to\theta_n$, only the zeros of Δ_n will merge. It can be confirmed that when $-\theta_n-\theta_0\geqslant\delta_1$, $\delta_1>0$, the stationary points are separated by a uniform distance from the zeros of Δ_n and the asymptotic expansions (3.2) are uniform in $\gamma:\delta\leqslant\gamma\leqslant\pi-\delta$.

Let us write $\omega_n=-\theta_n-\theta_0$ and let $|\omega_n|\ll 1$, i.e. we consider the region of the leading front of the n-th mode. The asymptotic form of the integrals (2.6) has, in this case, the following form with an accuracy of up to $O(R^{-\infty})$:

$$\begin{bmatrix} w_n \\ \zeta_n \end{bmatrix}\sim S_{n\nu}(R)+Z_{n\nu}(R,\theta_n)+Y_{n\nu}(R) \tag{4.1}$$

The last term in this formula represents the contribution of the near points θ_0 and $-\theta_n$:

$$Y_{n\nu}(R) = \mathrm{Re}\int_{-\pi/2}^{\pi/2} \Phi_{n\nu}\eta(\theta, \theta_0, -\theta_n)\, E_\nu(-R\Delta_n)\, d\theta \tag{4.2}$$

Here $\eta(\theta, \theta_0, -\theta_n) = \eta(\theta, \theta_0) + \eta(\theta, -\theta_n)$ is an infinitely differentiable finite function equal to unity on the segments whose ends coincide with the points θ_0 and $-\theta_n$.

Let us find the asymptotic form of the integral (4.2) when $R \to +\infty$. The substitution $\theta = -\theta_n - u^2$, $u = (-\theta_n - \theta)^{1/2}$ (the principal branch of the root is chosen) regularizes Δ_n and the integrand in (4.2) when $\nu = 1$. The expression for Δ_n can now be written in the form

$$\Delta_n(\theta, \gamma) = S_n(u, \omega_n),\ S_n(u, \omega_n) = up_n(u^2)\sin(\omega_n - u^2)$$

where $p_n(v) = \sqrt{r_n^2(\theta_n + v)/v}$ is a holomorphic function at the point $v = 0$. This follows from the fact that the dispersion relations $\beta_n(\lambda)$ of problem (1.4) are holomorphic functions $r_n^2(\theta_n) = 0$ and $dr_n^2(\theta_n)/d\theta > 0$. It can be confirmed that the function $S_n(u, \omega_n)$ satisfies all conditions of the lemmas 6.2.1-3 of /9/. Using these lemmas and remembering that $S_n(u, \omega_n)$ is an odd function of the variable u and returning after this to the initial variable θ, we have:

1°. When $|\omega_n| \neq 0$ are small, the equation $d\Delta_n/d\theta = 0$ has a single solution $\theta_{1n}(\omega_n)$, $\theta_{1n}(0) = -\theta_n$. The function $\theta_{1n}(\omega_n)$ is holomorphic for small ω_n and

$$\theta_{1n}(\omega_n) = -\theta_n - \frac{\omega_n}{3}\left(1 + \sum_{k=1}^{\infty} b_k\omega_n^k\right)$$

2°. A function $u_1(t, \omega)$, exists, holomorphic in (t, ω) in some neighbourhood of the point $(0, 0)$, such that $u_1(0, \omega_n) = -\theta_n$. After the substitution $\theta = u_1(\xi^2, \omega_n)$ the function $\Delta_n(\theta, \gamma)$ takes the form

$$\begin{gathered}\Delta_n(\theta, \gamma) = -S_0(\xi, B),\ S_0(\xi, B) = \xi^3/3 - \xi B \\ B = B(\omega_n),\ B(\omega_n) = r_n^2\,[3\sin(\theta - \gamma)/(r_n^2)_\theta']_{\theta=\theta_{1n}}^{2/3}\end{gathered} \tag{4.3}$$

The arithmetical branch of the root is chosen in the expression for $B(\omega_n)$.

In the reverse substitution $\xi = (-\theta_n - \theta)^{1/2}\zeta_1(-\theta_n - \theta, \omega_n)$, the function $\zeta_1(\tau, \omega)$ is holomorphic in (τ, ω) in some neighbourhood of the point $(0, 0)$.

3°. The function $B(\omega_n)$ is holomorphic at the point $\omega_n = 0$, and

$$B(\omega_n) = \omega_n\left[\frac{1}{3}\frac{dr_n^2(\theta_{1n})}{d\theta}\right]^{1/3}[1 + O(\omega_n)]$$

The substitution (4.3) transforms the integral (4.2) to the form

$$\begin{gathered}Y_{n\nu}(R) = \mathrm{Re}\int_{\infty}^{i\infty} \xi^{1-\nu}\varphi_{n\nu}(\xi^2)\, E_\nu[RS_0(\xi, B)]\, d\xi \\ \varphi_{n\nu}(\xi^2) = \xi^{\nu-1}\Phi_{n\nu}\eta(\theta, \theta_0, -\theta_n)\, d\theta/d\xi\end{gathered} \tag{4.4}$$

Wherever the functions of real variable are not defined, they are continued by means of the zeros. It can be confirmed that $\varphi_{n\nu}(t) \in C^\infty$.

We find the asymptotic forms of the integral (4.4) by integrating by parts and using (2.3). In the first stage we have

$$\begin{gathered}Y_{n\nu}(R) = \varphi_{n\nu}(B)\, J_{1-\nu}(R, B) - \nu R^{-1}\int_{\infty}^{i\infty} \xi D_1\varphi_{n\nu}(\xi^2)\ln|S_0|\, d\xi + \\ (-1)^\nu R^{-1}\int_{\infty}^{i\infty} \xi^\nu D_\nu\varphi_{n\nu}(\xi^2)\, E_{1-\nu}(RS_0)\, d\xi \\ J_{1-\nu}(R, B) = \mathrm{Re}\int_{\infty}^{i\infty} \xi^{1-\nu}E_\nu(RS_0)\, d\xi,\quad D_0\varphi(\xi^2) = (N + \xi^2 D_1)\,\varphi(\xi)^2 \\ N\varphi(\xi^2) = [\varphi(\xi^2) - \varphi(B)]\,/\,(\xi^2 - B),\ D_1\varphi(\xi^2) = \\ 2d\,[N\varphi(\xi^2)]/d\xi^2\end{gathered} \tag{4.5}$$

The last integral in (4.5) is of the same type as the initial integral. This enables us to use the recurrence methods to obtain the expansions of the integrals (4.4) as $R \to +\infty$. As a result we have

$$\begin{gathered}Y_{n\nu}(R) \sim -\sum_{m=0}^{\infty}(-1)^m R^{\nu-2m-2}\int_{\infty}^{i\infty} \xi D_1^\nu(D_{1-\nu}D_\nu)^m\varphi_{n\nu}(\xi^2)\ln|S_0|\, d\xi + \\ \sum_{m=0}^{\infty}(-1)^m R^{-2m}[J_{1-\nu}(R, B) + (-1)^\nu R^{-1}J_\nu(R, B)\, D_\nu] \times (D_{1-\nu}D_\nu)^m\,\varphi_{n\nu}(B)\end{gathered} \tag{4.6}$$

Let us evaluate the integrals $J_{1-\nu}(R, B)$. From (2.5) it follows that

$$J_{1-\nu}(R, B) = (-1)^\nu I_{1-\nu}(R, B) + \pi \operatorname{Im}\left\{i^\nu \int\limits_{\infty}^{i\infty} \xi^{1-\nu} \exp[iRS_0(\xi, B)]\, d\xi\right\}$$

$$I_{1-\nu}(R, B) = \operatorname{Re} \int\limits_{\infty}^{i\infty} \xi^{1-\nu} E_\nu[-RS_0(\xi, B)]\, d\xi$$

The functions $H_\nu(\xi) = E_\nu[-RS_0(\xi, B)]$ are holomorphic in the region $0 < \arg\xi < \pi/2$, and since $H_\nu(\xi) = O(\xi^{-6+3\nu})$ as $|\xi| \to \infty$ which follows from (2.4), it can be shown that the integrals $I_{1-\nu}(R, B)$ are equal to zero. The remaining single integrals can be expressed, after the substitution $\xi = R^{-1/3} t$, in terms of the Airy function /8/ and its derivative as

$$\begin{aligned} J_0(R, B) &= \pi^2 R^{-2/3} \operatorname{Ai}'(-BR^{2/3}) \\ J_1(R, B) &= -\pi^2 R^{-1/3} \operatorname{Ai}(-BR^{-2/3}) \end{aligned} \tag{4.7}$$

Thus the formulas (4.1), (4.6) and (4.7) together with (3.3) and (3.6) show that the asymptotic expansions, as $R \to +\infty$, of the integrals (2.6) uniform in γ near the leading front $\gamma = \gamma_n$, are given in the form of the sums of power series and the series containing Airy functions and their derivatives. Let us write out the principal terms of the asymptotic expansions when $|\gamma - \gamma_n| \ll 1$

$$\begin{aligned} w_n &= -\pi^2 R^{-2/3} \operatorname{Ai}'(-BR^{2/3}) \Phi_n \sqrt{-2/B^{1/2}\Delta''_{n\theta\theta}}\,\Big|_{\theta=\theta_{1n}} + O(R^{-4/3}) \\ \zeta_n &= \pi^2 R^{-1/3} \operatorname{Ai}(-BR^{2/3}) \Phi_n (r_n \cos\theta)^{-1} \sqrt{-2B^{1/2}/\Delta''_{n\theta\theta}}\,\Big|_{\theta=\theta_{1n}} + O(R^{-1}) \end{aligned} \tag{4.8}$$

On moving away from the leading front into the wave zone, when the argument of the Airy unction becomes large, the substitution of the asymptotic forms Ai and Ai'/8/ into (4.8), yields the principal terms of the asymptotic expansions of the contributions from the simple stationary point θ_{1n}.

We treat the second case of merging the singularities when $\theta_0 \to \theta_n$, in exactly the same manner. Here the formulas corresponding to (4.6) contain, instead of the integrals $J_\nu(R, B)$, the integrals $I_\nu(R, B)$ and the asymptotic form of the contribution of the near points θ_0 and θ_n is represented by a power series only.

The principal terms (4.8) of the asymptotic form of the contribution of the n-th mode in the neighbourhood of its leading front are analogous, when $c > c_n$, to those first obtained in /6/. The different result obtained in /5/ can be explained by the fact that in /5/ the authors dealt with the limiting case of a thin thermal wedge and the depth of the fluid tended to infinity. In the case of such an approximation the dispersion relation connecting the frequency ω with the wave number r can be expanded, for small values of r, in the series

$$\omega = c_0 r + br^2 + \ldots, \quad c_0, \quad b = \text{const}, \quad b \neq 0$$

For a fluid of finite depth, the dispersion relations $\omega_n(r)$ can be expanded in series in odd powers of r. Since the main contribution to (2.6) is given in the neighbourhood of the leading front of the n-th mode by the integral corresponding to small values of the wave number, the difference noted above appears to be a fundamental one.

In conclusion we note that the vertical displacements $\eta(x, y, z)$ caused by the action of a point dipole oriented against the flow, of moment d, are connected with the vertical displacements $\zeta(x, y, z)$, caused by the action of a point source of strength q by a simple formula: $\eta = dq^{-1}\partial\zeta/\partial x$. Comparing this formula with the kinematic relation $w = c\partial\zeta/dx$ we find that $\eta = d(cq)^{-1}w$ and the asymptotic formulas for w obtained in this paper differ only by a constant factor from the formulas for the field of vertical displacements caused by the action of a dipole. The cases of merger of the singularities discussed above when $\theta_0 \to \pm\theta_n$, occur only when $c > c_n$. The asymptotic expansions of the integrals (2.6) for the case $c < c_n$, were discussed in /10/.

REFERENCES

1. KELLER J.B. and MUNK W.H., Internal wave wakes of body moving in stratified fluid. Phys. Fluids, 13, 6, 1970.
2. MILES J.W., Internal waves generated by a horizontally moving source. Geophys. Fluid Dyn., 2, 1, 1970.
3. STUROVA I.V. and SUKHAREV V.A., Generation of internal waves by local perturbations in a fluid with a prescribed denisty variation with depth. Izv. AS SSSR, Fizika atmosfery i okeana, 17, 6, 1981.

4. SANNIKOV V.F. and CHERKESOV L.V., Development of spatial internal waves in a flow of stratified fluid. In: Internal and Surface Waves. Sevastopol, Izd. Mor. gidrofizich. in-ta Akad. Nauk SSSR, 1981.
5. GRAY E.P., HART R.W. and FARELL R.A., The structure of the internal wave Mach front generated by a point source moving in a stratified fluid. Phys. Fluids, 26, 10, 1983.
6. BOROVIKOV V.A., VLADIMIROV YU.V. and KEL'BERT M.YA., The internal gravitational wave field excited by localized sources. Izv. Akad. Nauk SSSR, Fizika atmosfery i okeana, 20, 6, 1984.
7. SANNIKOV V.F., The near field of stationary waves generated by a local perturbation source in a flow of stratified fluid. In: Theoretical Studies of Wave Processes in Ocean, Sevastopol, Izd, Mor. gidrofizich. in-ta, Akad. Nauk SSSR, 1983.
8. ABRAMOVITZ M. and STEGUN I.A., (Ed.). Handbook of Mathematical Functions. N.Y., Dover, 1956.
9. FEDORYUK M.V., The Saddle Point Method. Moscow, Nauka, 1977.
10. SANNIKOV V.F., Internal stationary waves generated by a local perturbation source within a flow, In: Modelling of Internal and Surface Waves. Sevastopol, Izd. Mor. gidrofizich. in-ta, Akad. Nauk SSSR, 1984.

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DIFFUSION ANALOGUE OF A COMBUSTION WAVE IN A SYSTEM WITH DISCRETE SOURCES*

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The problem of selfsustaining concentration waves in discrete quasione-dimensional system with diffusion and threshold activation of the sources is considered. A number of applications of such models for describing spontaneous contraction waves observed in the course of experiments involving single muscle cells is discussed.

For many bological objects the passive transport of matter (caused by a difference in concentration) across some barriers such as biological membranes, depends in a complex manner on the absolute values of the concentrations on both sides of the barrier. Normally this is caused by the fact that transported material takes part in the chemical reactions which radically alter the effective permeability of the barrier.

One of the most interesting processes of this type is the release of Ca^{2+} ions from the inner cavities (terminal cisterns) of the cardiac musle cell directly into the contractile apparatus, the release occurring when the concentration of these ions outside the cavities reaches some threshold value. It is also necessary, in order for the release to occur, that this external concentration should approach its threshold value from below and at a sufficently rapid rate /1, 2/. The membrane which confines the intracellular cisterns, distributed within the cells in an orderly manner and separated from each other by distances of order at least equal to the size of the cisterns themselves, is regarded as the barrier.

The successive release of the ions from the cisterns can occur either as the result of diffusion of the released calcium, or with the help of electric control signals /3/ propagating rapidly along the cell. The signal can appear, in principle, as a result of large changes in ion concentrations occurring after calcium has been released from the cisterns. The release of calcium from the sequentially distributed cisterns results in the formation of a wave of increased concentration of calcium propagating along the cell, followed by a mechanical concentration wave which was observed experimentally in /4/ (the Ca^{2+} ions locally trigger the performance of contractile structures of the cell).

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